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q-Bernstein polynomials and their iterates

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Abstract

Let $B_n(f,q;x),\ n=1,2,\dots$ be q-Bernstein polynomials of a function $f:[0,1]\to\mathbf{C}$. The polynomials $B_n(f,1;x)$ are classical Bernstein polynomials. For $q\neq 1$ the properties of q-Bernstein polynomials differ essentially from those in the classical case. This paper deals with approximating properties of q-Bernstein polynomials in the case $q>1$ with respect to both n and q. Some estimates on the rate of convergence are given. In particular, it is proved that for a function f analytic in $\{z\colon |z|<q+\varepsilon\}$ the rate of convergence of $\{B_n(f,q;x)\}$ to $f(x)$ in the norm of $C[0,1]$ has the order q^{-n} (versus $1/n$ for the classical Bernstein polynomials). Also iterates of q-Bernstein polynomials $\{B_n^{j_n}(f,q;x)\}$, where both $n\to\infty$ and $j_n\to\infty$, are studied. It is shown that for $q\in(0,1)$ the asymptotic behavior of such iterates is quite different from the classical case. In particular, the limit does not depend on the rate of $j_n\to\infty$.

Keywords: q-Bernstein polynomials; q-Integers; q-Binomial coefficients; Convergence; Iterates

1. Introduction

In 1912 Bernstein [2] found his famous proof of the Weierstrass Approximation Theorem. Using probability theory he defined polynomials called nowadays *Bernstein polynomials* as follows.

Definition (Bernstein [2]). Let $f:[0,1]\to\mathbf{R}$. The *Bernstein polynomial* of f is

$$B_n(f;x) := \sum_{k=0}^{n} f\left(\frac{k}{n}\right)\binom{n}{k}x^k(1-x)^{n-k},\quad n=1,2,\dots\ .$$

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Bernstein proved that if $f \in C[0,1]$, then the sequence $\{B_n(f;x)\}$ converges uniformly to $f(x)$ on $[0,1]$.

Later it was found that Bernstein polynomials possess many remarkable properties, which made them an area of intensive research. A systematic treatment of the theory of Bernstein polynomials as it was until 1990s is presented, for example, in [8,19]. New papers are constantly coming out (cf. e.g. [3]), and new applications and generalizations are being discovered (cf. e.g. [7,13]). A generalization of Bernstein polynomials involving q-integers was proposed by Lupaş in 1987 (cf. [9]). However, the q-analogue of the Bernstein operator considered by Lupaş gives rational functions rather than polynomials.

Generalized Bernstein polynomials based on the q-integers, or *q-Bernstein polynomials* were introduced by Phillips in 1997. In the case $q = 1$ these polynomials coincide with the classical ones. For $q \neq 1$ one gets a new class of polynomials having interesting properties. q-Bernstein polynomials have been studied by Phillips et al. ([4,11,12,14–17]), who obtained a great number of results related to various properties of these polynomials.

It should be mentioned that results of these papers deal mostly with the case $q \in (0,1)$. This is because in this case q-Bernstein polynomials generate *positive* linear operators $B_{n,q} : f \mapsto B_n(f,q;x)$; the fact that is used in investigation significantly. The case $q \in (1,\infty)$, where positivity fails, has not been studied in detail. However, the results of this paper show that in this case approximating properties of q-Bernstein polynomials may be better than in the case $q \leqslant 1$.

In Sections 3 and 4, we discuss convergence properties of q-Bernstein polynomials with respect to both n and q in the case $q > 1$.

In Sections 5 and 6, we study the rate of approximation of analytic functions by q-Bernstein polynomials in the case $q > 1$. In particular, for entire functions the rate of convergence has the order q^{-n} $(q > 1)$ versus $1/n$ for the classical polynomials. We also discuss approximation by q-Bernstein polynomials in case the value of parameter q varies.

It should be emphasized that the results of the paper are the first ones showing that approximation properties of q-Bernstein polynomials can be *better* than of the classical ones.

Sections 7–9 are dedicated to iterates of the q-Bernstein operator. By the definition the kth iterate of $B_{n,q}$ is

$$B^1_{n,q} := B_{n,q}, \quad B^k_{n,q} := B_{n,q}(B^{k-1}_{n,q}), \quad k = 2, 3, \ldots .$$

Iterates of the classical Bernstein operator $B_n := B_{n,1}$ have been studied in many papers starting from [6]. In [6], Kelisky and Rivlin studied the convergence of the iterates $B^k_n(f)$ as $k \to \infty$ if n is fixed, and of the iterates $B^{j_n}_n(p)$ as $n \to \infty$, where p is a polynomial and $\{j_n/n\} \to \alpha$, $0 \leqslant \alpha \leqslant \infty$. They proved that in both cases the iterates are convergent, and found an explicit formula of the limit function. From a different point of view the iterates of the Bernstein operator were studied by Micchelli [10], who considered them using semigroup methods. Recently, Cooper and Waldron [3] investigated iterates of the Bernstein operator using properties of eigenvalues and eigenvectors of the operator. In [3] one can also find other references on the subject.

Iterates of the q-Bernstein operator $B^k_{n,q}$ with fixed n and $k \to \infty$ were considered in [12], where it was proved that these iterates have the same behavior as in the classical case $q = 1$.

In this paper we consider iterates of the q-Bernstein operator of the form $B^{j_n}_{n,q}$, where both n and j_n tend to infinity. We consider in detail the behavior of iterates of the q-Bernstein operator for $q \in (0, 1)$. Our results show that in this case the behavior of iterates is essentially different from the classical case $q = 1$ considered by Kelisky and Rivlin [6, Theorem 2]. In particular, the limit does not depend on the rate of $j_n \to \infty$ (cf. Theorem 8). For $q \in (1, \infty)$ the situation is very similar to the classical case. Corresponding results and their proofs can be obtained by almost verbatim extension of reasoning given in [3, Theorems 4.1, 4.20, Corollary 5.15]. Therefore, we present them without proofs.

To formulate our results we need the following definitions.

Let $q > 0$. For any $n = 0, 1, 2, \ldots$ the *q-integer* $[n]_q$ is defined by

$$[n]_q := 1 + q + \cdots + q^{n-1} \ (n = 1, 2, \ldots), \ [0]_q := 0$$

and the *q-factorial* $[n]_q!$ by

$$[n]_q! := [1]_q[2]_q \ldots [n]_q \ (n = 1, 2, \ldots), \ [0]_q! := 1.$$

For integers $0 \leqslant k \leqslant n$ the *q-binomial,* or the Gaussian coefficient is defined by

$$\begin{bmatrix} n \\ k \end{bmatrix}_q := \frac{[n]_q!}{[k]_q![n-k]_q!}.$$

Clearly, for $q = 1$,

$$[n]_1 = n, \quad [n]_1! = n!, \quad \begin{bmatrix} n \\ k \end{bmatrix}_1 = \binom{n}{k}.$$

In the sequel we always assume that $f : [0, 1] \to \mathbf{C}$. We denote by $C[0, 1]$ (or $C^n[0, 1]$, $1 \leqslant n \leqslant \infty$) the space of all continuous (correspondingly, n times continuously differentiable) complex-valued functions on [0,1] equipped with the uniform norm. The expression $g_n(x) \rightrightarrows g(x)$ means uniform convergence of a sequence $\{g_n(x)\}$ to $g(x)$.

Definition (Phillips [14]). Let $f : [0, 1] \to \mathbf{C}$, $q > 0$. The *q-Bernstein polynomial* of f is

$$B_n(f, q; x) := \sum_{k=0}^{n} f\left(\frac{[k]_q}{[n]_q}\right) \begin{bmatrix} n \\ k \end{bmatrix}_q x^k \prod_{s=0}^{n-1-k} (1 - q^s x), \quad n = 1, 2, \ldots \ . \tag{1}$$

(From here on an empty product is taken to be equal 1.)

Note that for $q = 1$, the polynomials $B_n(f, 1; x)$ are classical Bernstein polynomials. Recall that the famous theorem of Bernstein states:

Theorem (Bernstein [2]). *If* $f \in C[0,1]$, *then*

$$B_n(f,1;x) \rightrightarrows f(x) \quad \textit{for } x \in [0,1] \textit{ as } n \to \infty.$$

For $q \in (0,1)$ convergence of the sequence $\{B_n(f,q;x)\}$ was investigated in [5].

Theorem (Il'inskii and Ostrovska [5]). *Given* $q \in (0,1)$ *and* $f \in C[0,1]$, *there exists a continuous function* $B_\infty(f,q;x)$ *such that*

$$B_n(f,q;x) \rightrightarrows B_\infty(f,q;x) \quad \textit{for } x \in [0,1] \textit{ as } n \to \infty. \tag{2}$$

An explicit formula for $B_\infty(f,q;x)$ is given by (16). It follows from (16) that the equality $B_\infty(f,q;x) = f(x)$ holds if and only if $f(x) = ax + b$, i.e. $f(x)$ is a linear function.

Therefore, in the case $q \in (0,1)$ the sequence $\{B_n(f,q;x)\}$ is not an approximating sequence for a function f unless f is linear. This is in contrast to the case $q = 1$, when the sequence $\{B_n(f,1;x)\}$ approximates f for any $f \in C[0,1]$.

In this paper we show that in the case $q > 1$ approximating properties of the sequence $\{B_n(f,q;x)\}$ are in some sense intermediate between the cases mentioned above. We prove that for $q > 1$ the sequence $\{B_n(f,q;x)\}$ is approximating for functions analytic in a suitable domain, and, moreover, we may achieve a fast rate of convergence. At the same time the sequence may be divergent for some infinitely differentiable functions. We also discuss approximating properties of q-Bernstein polynomials related to the dependence on the value of q.

Equality (1) defines the linear operator

$$B_{n,q} : f \mapsto B_n(f,q;x),$$

which is called the *q-Bernstein operator*. Clearly,

$$B_{n,q} : C[0,1] \to \mathscr{P}_n,$$

where $\mathscr{P}_n$ denotes the set of polynomials of degree $\leqslant n$. To study iterates of q-Bernstein polynomials it is convenient to present them in the form of linear operators, i.e. $B_n^k(f,q,x) = B_{n,q}^k(f)$. In the sequel, we use polynomial and operator notation interchangeably. We prove that for $q \in (0,1)$ and any function $f \in C[0,1]$ the sequence $\{B_n^{j_n}(f,q,x)\}$, where $n \to \infty$ and $j_n \to \infty$, converges uniformly to the linear function interpolating f at 0 and 1 regardless the rate of $j_n \to \infty$.

For $q \in (0,1)$ the limit function appeared in (2) defines a linear operator on $C[0,1]$

$$B_{\infty,q} : f \mapsto B_\infty(f,q;x).$$

It was observed in [5] that $B_{\infty,q}(C[0,1]) \neq C[0,1]$. We also consider the behavior of the iterates of $B_{\infty,q}$.

2. Preliminaries

In this section we state some general properties of q-Bernstein polynomials which will be used throughout the paper.

It follows directly from the definition that q-Bernstein polynomials possess the *end-point interpolation* property, i.e.

$$B_n(f,q;0)=f(0),\ B_n(f,q;1)=f(1)\ \text{for all}\ q>0\ \text{and all}\ n=1,2,\dots\,. \tag{3}$$

The following representation of q-Bernstein polynomials, called the *q-difference form*, was obtained in [15, Theorem 1, formula (12)]:

$$B_n(f,q;x)=\sum_{k=0}^{n}\begin{bmatrix} n\\ k\end{bmatrix}_q \mathscr{D}^k f_0\, x^k, \tag{4}$$

where $\mathscr{D}^k f_0$ is expressed as

$$\mathscr{D}^k f_0=\frac{[k]_q!}{[n]_q^k}q^{k(k-1)/2}f\left[0;\frac{1}{[n]_q};\cdots;\frac{[k]_q}{[n]_q}\right]. \tag{5}$$

By $f[x_0;x_1;\dots;x_k]$ we denote the usual divided difference, i.e.

$$f[x_0]=f(x_0),\quad f[x_0;x_1]=\frac{f(x_1)-f(x_0)}{x_1-x_0},\dots,$$

$$f[x_0;x_1;\dots;x_j]=\frac{f[x_1;\dots;x_j]-f[x_0;\dots;x_{j-1}]}{x_j-x_0}.$$

Using (4) and (5), we write

$$B_n(f,q;x)=\sum_{k=0}^{n}\lambda_{k,q}^{(n)}f\left[0,\frac{1}{[n]_q},\dots,\frac{[k]_q}{[n]_q}\right]x^k, \tag{6}$$

where

$$\lambda_{k,q}^{(n)}:=\begin{bmatrix} n\\ k\end{bmatrix}_q\frac{[k]_q!}{[n]_q^k}q^{k(k-1)/2}=\left(1-\frac{1}{[n]_q}\right)\cdots\left(1-\frac{[k-1]_q}{[n]_q}\right). \tag{7}$$

In Section 8 (Lemma 5) we show that $\lambda_{k,q}^{(n)}$ are eigenvalues of the q-Bernstein operator $B_{n,q}$. Note that

$$\lambda_{0,q}^{(n)}=\lambda_{1,q}^{(n)}=1, \tag{8}$$

and it is clear from (7) that

$$0\leqslant\lambda_{k,q}^{(n)}\leqslant 1,\quad k=0,1,\dots,n. \tag{9}$$

Therefore,

$$|B_n(f,q;x)|\leqslant\sum_{k=0}^{n}\left|f\left[0,\frac{1}{[n]_q},\dots,\frac{[k]_q}{[n]_q}\right]\right||x|^k. \tag{10}$$

This estimate will be used in the sequel.

It follows immediately from (6) and (8) that q-Bernstein polynomials leave invariant linear functions, that is

$$B_n(at+b,q;x)=ax+b\quad\text{for all}\ q>0\ \text{and all}\ n=1,2\dots\,. \tag{11}$$

If f is a polynomial of degree m, then all its divided differences of order $>m$ vanish, and (6) implies that $B_n(f,q;x)$ is a polynomial of degree $\min(m,n)$. In other words, this means that the q-Bernstein operator is degree reducing.

We set

$$p_{nk}(q;x) := \begin{bmatrix} n \\ k \end{bmatrix}_q x^k \prod_{s=0}^{n-1-k} (1-q^s x), \quad k=0,1,\dots,n; \; n=1,2,\dots \; . \tag{12}$$

Taking $a=0$, $b=1$ in (11), we conclude that

$$\sum_{k=0}^{n} p_{nk}(q;x) = 1; \quad \text{for all } q>0 \text{ and all } n=1,2,\dots \; . \tag{13}$$

Obviously,

$$B_n(f,q;x) = \sum_{k=0}^{n} f\left(\frac{[k]_q}{[n]_q}\right) p_{nk}(q;x).$$

The behavior of the sequence $\{B_n(f,q;x)\}$ for $q\in(0,1)$ and $n\to\infty$ is described in [5] as follows.

Consider the entire functions

$$p_{\infty k}(q;x) := \frac{x^k}{(1-q)^k [k]_q!} \prod_{s=0}^{\infty} (1-q^s x), \quad k=0,1,\dots \; . \tag{14}$$

By Euler's identity (cf. [1, Chapter 2, Corollory 2.2]) we have

$$\sum_{k=0}^{\infty} p_{\infty k}(q;x) = 1 \quad \text{for all } x\in[0,1). \tag{15}$$

Clearly, for $q\in(0,1)$ we have

$$\lim_{n\to\infty} \frac{[k]_q}{[n]_q} = 1-q^k \quad \text{for all } k=0,1,\dots \; .$$

For $f:[0,1]\to\mathbf{C}$, $q\in(0,1)$ we set

$$B_\infty(f,q;x) = \begin{cases} \sum_{k=0}^{\infty} f(1-q^k) p_{\infty k}(q;x) & \text{if } x\in[0,1), \\ f(1) & \text{if } x=1. \end{cases} \tag{16}$$

It can be readily seen that the function $B_\infty(f,q;x)$ is well defined on $[0,1]$ whenever a function $f(x)$ is bounded on the interval. We note that (16) gives the limit function defined in (2). It follows from (2) and (11) that

$$B_\infty(at+b,q,x) = ax+b. \tag{17}$$

In the following section we investigate the behavior of the sequence $\{B_n(f,q;x)\}$ in the case $q>1$.

3. Convergence of q-Bernstein polynomials in the case $q>1$

Our main result on convergence is the following theorem.

Theorem 1. *Let $q \in (1, \infty)$, and let f be a function analytic in an ε-neighborhood of* [0,1]. *Then for any compact set $K \subset D_\varepsilon := \{z: |z| < \varepsilon\}$,*

$$B_n(f, q; z) \rightrightarrows f(z) \quad \text{for } z \in K \text{ as } n \to \infty.$$

Corollary 1. *If f is a function analytic in a disk D_R, $R>1$, then for any compact set $K \subset D_{R-1}$,*

$$B_n(f, q; z) \rightrightarrows f(z) \quad \text{for } z \in K \text{ as } n \to \infty.$$

In particular, if $R>2$, then $B_n(f, q; x) \rightrightarrows f(x)$ for $x \in [0, 1]$ as $n \to \infty$.

Corollary 2. *If f is an entire function, then for any compact set $K \subset \mathbf{C}$,*

$$B_n(f, q; z) \rightrightarrows f(z) \quad \text{for } z \in K \text{ as } n \to \infty.$$

Remark. A particular case f being a polynomial and $K = [0, 1]$ was considered in [12].

The condition of analyticity is essential for convergence, and it cannot be dropped completely as the following theorem shows.

Theorem 2. *Let $q \in (1, \infty)$.*

(i) *There exists $f \in C^\infty[0, 1]$ such that $\{B_n(f, q; x)\}$ does not converge to any finite function on $[0, 1]$.*
(ii) *There exists $f \in C^\infty[0, 1]$ such that $\{B_n(f, q; x)\}$ converges to a finite discontinuous function on $[0, 1]$.*
(iii) *There exists $f \in C^\infty[0, 1]$ such that $\{B_n(f, q; x)\}$ converges uniformly on $[0, 1]$ to $g(x) \neq f(x)$.*

The following theorem describes the behavior of the polynomials $B_n(f, q; x)$ as $q \to +\infty$ under certain smoothness conditions for f.

Theorem 3. *Let $f \in C^{n-1}[0, 1]$. Then for any compact set $K \subset \mathbf{C}$,*

$$B_n(f, q; z) \rightrightarrows B_n(f, \infty; z) := \sum_{k=0}^{n-1} \frac{f^{(k)}(0)}{k!} z^k + z^n \left\{ f(1) - \sum_{k=0}^{n-1} \frac{f^{(k)}(0)}{k!} \right\}$$

for $z \in K$ as $q \to +\infty$.

Corollary 3. *If f is analytic in a disk D_R, $R>1$, then*

$$B_n(f,\infty;z)\rightrightarrows f(z) \quad \text{for } |z|\leqslant 1 \text{ as } n\to\infty.$$

That is, quite unexpectedly, we get good approximating properties of the sequence $\{B_n(f,q;x)\}$ taking the value of q infinite. The corollary below can be derived from Theorem 3 immediately.

Corollary 4. *If p is a polynomial of degree $\leqslant n$, then*

$$B_n(p,\infty;z)=p(z).$$

Therefore, we may approximate $p(x)$ with its q-Bernstein polynomials of the same degree n taking the limit with respect to q.

4. Proofs of Theorems 1–3

We need the following lemma, which is also of interest for its own sake.

Lemma 1. *Let $q\in(1,\infty)$. If $f\in C[0,1]$, then*

$$\lim_{n\to\infty} B_n\left(f,q;\frac{1}{q^m}\right)=f\left(\frac{1}{q^m}\right) \quad \text{for all } m=0,1,2,\ldots\,.$$

Proof. Let the polynomials $p_{nk}(q;x)$ be defined by (12). Obviously,

$$B_n(f,q;x)=\sum_{k=0}^{n} f\left(\frac{[n-k]_q}{[n]_q}\right)p_{n,n-k}(q;x).$$

We note that

$$p_{n,n-k}\left(q;\frac{1}{q^m}\right)=0 \quad \text{for } m<k\leqslant n$$

and

$$p_{n,n-k}\left(q;\frac{1}{q^m}\right)=\begin{bmatrix} n \\ k \end{bmatrix}_q \frac{1}{q^{m(n-k)}}\left(1-\frac{1}{q^m}\right)\cdots\left(1-\frac{q^{k-1}}{q^m}\right)$$
$$=O(q^{n(k-m)})\to 0 \quad \text{as } n\to\infty \text{ for } k<m.$$

For $k=m$ we have

$$\lim_{n\to\infty} p_{n,n-m}\left(q;\frac{1}{q^m}\right)=\lim_{n\to\infty}\begin{bmatrix} n \\ n-m \end{bmatrix}_q \frac{1}{q^{m(n-m)}}\left(1-\frac{1}{q^m}\right)\cdots\left(1-\frac{1}{q}\right)=1.$$

Since $f\in C[0,1]$ and

$$\lim_{n\to\infty}\frac{[n-m]_q}{[n]_q}=\frac{1}{q^m},$$

it follows that

$$\lim_{n\to\infty} B_n\left(f, q; \frac{1}{q^m}\right) = \lim_{n\to\infty} f\left(\frac{[n-m]_q}{[n]_q}\right) p_{n,n-m}\left(q; \frac{1}{q^m}\right) = f\left(\frac{1}{q^m}\right). \qquad \square$$

Proof of Theorem 1. Let f be analytic in an ε-neighborhood U_ε of $[0,1]$. Take any compact set $K \subset D_\varepsilon$. Then for some $\varepsilon_1 \in (0, \varepsilon)$ we have $|z| \leqslant \varepsilon_1$ for all $z \in K$.

Let us choose a contour L in U_ε in such a way that the distance between L and $[0,1]$ equals ρ, $0 < \varepsilon_1 < \rho < \varepsilon$.

Since (cf. [8, Chapter II, Section 2.7])

$$f\left[0, \frac{1}{[n]_q}, \dots, \frac{[k]_q}{[n]_q}\right] = \frac{1}{2\pi i} \int_L \frac{f(\zeta)\, d\zeta}{\zeta\left(\zeta - \frac{1}{[n]_q}\right)\cdots\left(\zeta - \frac{[k]_q}{[n]_q}\right)}$$

and $|\zeta - x| \geqslant \rho$ for all $\zeta \in L$ and $x \in [0,1]$, it follows that

$$\left| f\left[0, \frac{1}{[n]_q}, \dots, \frac{[k]_q}{[n]_q}\right] \right| \leqslant \frac{l}{2\pi} \cdot \frac{M_L}{\rho^{k+1}}, \tag{18}$$

where l is the length of L, and $M_L = \max_{\zeta \in L} |f(\zeta)|$. Substituting (18) into (10), we obtain

$$|B_n(f, q; z)| \leqslant \frac{l M_L}{2\pi\rho} \sum_{k=0}^{n} \frac{|z|^k}{\rho^k}.$$

If $z \in D_{\varepsilon_1}$, then $|z| < \varepsilon_1 < \rho$, so

$$\sum_{k=0}^{n} \frac{|z|^k}{\rho^k} \leqslant \sum_{k=0}^{n} \left(\frac{\varepsilon_1}{\rho}\right)^k < \sum_{k=0}^{\infty} \left(\frac{\varepsilon_1}{\rho}\right)^k = \frac{1}{1 - \frac{\varepsilon_1}{\rho}},$$

and hence the sequence $\{B_n(f, q; z)\}$ is uniformly bounded in the disk D_{ε_1}. Besides, by Lemma 1 the sequence converges to the function f analytic in D_{ε_1} on the set $\{1/q^m\}_0^\infty$ having an accumulation point in D_{ε_1}. By the Vitali Theorem (cf. e.g. [18, Chapter V, Section 5.2]) the sequence converges to f on any compact set in D_{ε_1}, and thus on K. $\square$

Proof of Theorem 2. Consider the polynomials $p_{nk}(q; x)$ defined by (12).

Specifically, we have

$$p_{nn}(q; x) = x^n$$

and

$$p_{n,n-1}(q; x) = [n]_q x^{n-1}(1 - x) = \frac{q^n - 1}{q - 1} x^{n-1}(1 - x).$$

Obviously,

$$\lim_{n\to\infty} p_{nn}(q; x) = \begin{cases} 0 & \text{for } 0 \leqslant x < 1, \\ 1 & \text{for } x = 1 \end{cases}$$

and

$$\lim_{n\to\infty} p_{n,n-1}(q;x) = \begin{cases} 0 & \text{for } 0\leqslant x<1/q \text{ and } x=1, \\ 1 & \text{for } x=1/q \\ \infty & \text{for } 1/q<x<1. \end{cases} \tag{19}$$

(i) Consider a function $\varphi \in C^{\infty}[0,1]$ satisfying

$$\varphi(x) = \begin{cases} 0 & \text{for } 0\leqslant x\leqslant 1/q^2, \\ 1 & \text{for } a\leqslant x\leqslant 1/q, \\ 0 & \text{for } x=1, \end{cases}$$

where $a\in(1/q^2, 1/q)$. Since

$$\frac{[n-k]_q}{[n]_q}\uparrow\frac{1}{q^k} \quad \text{as } n\to\infty,$$

we obtain that

$$\varphi\left(\frac{[n-k]_q}{[n]_q}\right) = 0 \quad \text{for } k\neq 1 \text{ and sufficiently large } n.$$

Therefore

$$B_n(\varphi, q; x) = \varphi\left(\frac{[n-1]_q}{[n]_q}\right) p_{n,n-1}(q;x) = p_{n,n-1}(q;x)$$

for n large enough.

Let $g(x)$ be an entire function. We set

$$f(x) := g(x) + \varphi(x).$$

Then $B_n(f,q;x) = B_n(g,q;x) + B_n(\varphi,q;x)$. By Theorem 1, $B_n(g,q;x)\rightrightarrows g(x)$ on $[0,1]$. Hence

$$\lim_{n\to\infty} B_n(f,q;x) = g(x) + \lim_{n\to\infty} p_{n,n-1}(q;x).$$

By (19) the limit is infinite for $x\in(1/q,1)$.

(ii) In this case we take $\varphi\in C^{\infty}[0,1]$ to satisfy

$$\varphi(x) = \begin{cases} 0 & \text{for } 0\leqslant x\leqslant 1/q, \\ 1 & \text{for } x=1. \end{cases}$$

Similar to (i) we take an entire function $g(x)$ and set $f(x) := g(x) + \varphi(x)$. Since $B_n(g,q;x)\rightrightarrows g(x)$ and $B_n(\varphi,q;x) = p_{nn}(q;x) = x^n$, we are done.

(iii) Consider $0\not\equiv\varphi(x)\in C^{\infty}[0,1]$ such that $\varphi(x)=0$ for $x\in[0,1/q]\cup\{1\}$. Obviously, $B_n(\varphi,q;x)\equiv 0$ for all $n=1,2,\dots$. For any entire function $g(x)$ we set as above $f(x) := g(x)+\varphi(x)$ and get $B_n(f,q;x)\rightrightarrows g(x)\neq f(x)$. $\square$

Proof of Theorem 3. Using (6) and (7) we write

$$B_n(f,q;z)=\sum_{k=0}^{n}\left(1-\frac{1}{[n]_q}\right)\cdots\left(1-\frac{[k-1]_q}{[n]_q}\right)f\left[0,\frac{1}{[n]_q},\ldots,\frac{[k]_q}{[n]_q}\right]z^k.$$

Note that for $j<n$,

$$\lim_{q\to+\infty}\frac{[j]_q}{[n]_q}=0,$$

so all factors in the parentheses tend to 1 as $q\to+\infty$.

Now, since $f\in C^{n-1}[0,1]$, for $k\leqslant n-1$ we get

$$\lim_{q\to+\infty}f\left[0,\frac{1}{[n]_q},\ldots,\frac{[k]_q}{[n]_q}\right]=\frac{f^{(k)}(0)}{k!}.$$

This allows us to evaluate the limit of the coefficients of $1,z,\ldots,z^{n-1}$ in $B_n(f,q;x)$ as $q\to+\infty$. To find the limit of the coefficient of z^n we must evaluate

$$\lim_{q\to+\infty}f\left[0,\frac{1}{[n]_q},\ldots,\frac{[n-1]_q}{[n]_q},1\right].$$

We will use the following lemma.

Lemma 2. *Let* $f\in C^m[0,1]$ *and* $0\leqslant x_0<x_1<\cdots<x_m<1$. *Then*

$$\lim_{x_m\to 0}f[x_0,x_1,\ldots,x_m,1]=f(1)-\sum_{k=0}^{m}\frac{f^{(k)}(0)}{k!}.$$

Proof. We prove the lemma by induction on m.

For $m=0$, we have

$$f[x_0,1]=\frac{f(1)-f(x_0)}{1-x_0}$$

and, clearly,

$$\lim_{x_0\to 0}f[x_0,1]=f(1)-f(0).$$

Assume that the statement is true if the number of points x_i does not exceed m. Consider the divided difference with $(m+1)$ points x_i:

$$f[x_0,x_1,\ldots,x_m,1]=\frac{f[x_1,\ldots,x_m,1]-f[x_0,x_1,\ldots,x_m]}{1-x_m}.$$

By the induction assumption we have

$$\lim_{x_m\to 0}f[x_1,\ldots,x_m,1]=f(1)-\sum_{k=0}^{m-1}\frac{f^{(k)}(0)}{k!}.$$

On the other hand, since $f \in C^m[0,1]$, we get

$$\lim_{x_m \to 0} f[x_0, x_1, \ldots, x_m] = \frac{f^{(m)}(0)}{m!}.$$

Thus,

$$\lim_{x_m \to 0} f[x_0, x_1, \ldots, x_m, 1] = f(1) - \sum_{k=0}^{m} \frac{f^{(k)}(0)}{k!}. \qquad \square$$

Applying Lemma 2 we obtain

$$\lim_{q \to +\infty} f\left[0, \frac{1}{[n]_q}, \ldots, \frac{[n-1]_q}{[n]_q}, 1\right] = f(1) - \sum_{k=0}^{n-1} \frac{f^{(k)}(0)}{k!}.$$

Finally, we get for $f \in C^{n-1}[0,1]$,

$$\lim_{q \to +\infty} B_n(f, q; z) = \sum_{k=0}^{n-1} \frac{f^{(k)}(0)}{k!} z^k + z^n \left\{ f(1) - \sum_{k=0}^{n-1} \frac{f^{(k)}(0)}{k!} \right\}. \qquad \square$$

5. Rate of convergence of q-Bernstein polynomials in the case $q > 1$

The following is a Voronovskaya-type theorem for monomials. It shows that in the case $q > 1$ the polynomials $B_n(t^m, q; z)$ converge to z^m essentially faster than the classical ones.

Theorem 4. *Let $q \geqslant 1$ be fixed. Then for any $z \in \mathbf{C}$,*

$$\lim_{n \to \infty} [n]_q \{B_n(t^m, q; z) - z^m\} = (1 + [2]_q + \cdots + [m-1]_q)(z^{m-1} - z^m).$$

(From here on an empty sum is taken to be equal 0.)

The following theorem provides a uniform estimate of the difference between z^m and its q-Bernstein polynomial in a circle of radius $R > 1$.

Theorem 5. *Let $q \geqslant 1$ be fixed. Then for $R > 1$ and all $m = 1, 2, \ldots;\ n = 1, 2, \ldots$ we have*

$$|B_n(t^m, q; z) - z^m| \leqslant 2 \frac{(m-1)[m-1]_q}{[n]_q} R^m \quad \textit{for } |z| \leqslant R.$$

Corollary 5. *Let $q \geqslant 1$ be fixed. Then for any compact set $K \subset \mathbf{C}$,*

$$B_n(t^m, q; z) \rightrightarrows z^m \quad \textit{for } z \in K \textit{ as } n \to \infty,$$

and

$$|B_n(t^m, q; z) - z^m| \leqslant \frac{C_{m,q,K}}{[n]_q} \quad \textit{for all } n = 1, 2, \ldots .$$

If we consider q-Bernstein polynomials of z^m in the closed unit disk $\{z\colon |z| \leqslant 1\}$, we can get a more particular estimate.

Corollary 6. *For all* $m = 0, 1, 2, \dots, n = 1, 2, \dots,$ *we have*

$$|B_n(t^m, q; z) - z^m| \leqslant 2 \frac{mq^m}{[n]_q(q-1)} \quad \text{for } |z| \leqslant 1.$$

Using the latter estimate we obtain the following statement, which shows that for a wide class of analytic functions their q-Bernstein polynomials provide exponentially fast approximation in the closed unit disk, and, in particular, on the interval $[0, 1]$.

Theorem 6. *Let* $q \geqslant 1$ *be fixed. If a function* $f(z)$ *is analytic in a disk* $D_R := \{z\colon |z| < R\}$, $R > q$, *then*

$$|B_n(f, q; z) - f(z)| \leqslant \frac{C_{f,q}}{[n]_q} \quad \text{for } |z| \leqslant 1 \text{ and all } n = 1, 2 \dots .$$

That is, if a function is analytic in a disk of radius $R > q$, then its q-Bernstein polynomials form an approximating sequence on [0,1] with the rate of convergence of order q^{-n}. Therefore, in the case $q > 1$ approximation of an analytic function with q-Bernstein polynomials is essentially faster than with the classical ones.

It turns out that in the case $q \geqslant 1$, q-Bernstein polynomials of an analytic function form an approximating sequence in the closed unit disk $\{z\colon |z| \leqslant 1\}$ even if we do not keep the value of q fixed.

Theorem 7. *If a function* $f(z)$ *is analytic in a disk* D_R, $R > 1$, *then for all* $q \geqslant 1$ *the following estimate holds uniformly with respect to* q:

$$|B_n(f, q; z) - f(z)| \leqslant \frac{C_f}{n} \quad \text{for } |z| \leqslant 1 \text{ and all } n = 1, 2 \dots .$$

The following corollary can be regarded as an analogue for $q_n \geqslant 1$ of Phillips' convergence theorem [15, Theorem 2].

Corollary 7. *If a function* $f(z)$ *is analytic in a disk* D_R, $R > 1$, *then for any sequence* $\{q_n\}$, $q_n \geqslant 1$ *we have*

$$B_n(f, q_n, z) \rightrightarrows f(z) \quad \text{for } |z| \leqslant 1 \text{ as } n \to \infty.$$

6. Proofs of Theorems 4–7

The following lemma is needed for the sequel.

Lemma 3. *Let $f = t^m$, $m \geqslant 1$. Then*

$$B_n(t^m, q; z) = \alpha_1 z + \cdots + \alpha_j z^j, \quad j = \min(m, n), \tag{20}$$

where
(i) *all* $\alpha_i \geqslant 0$ $(i = 1, \ldots, j)$.
(ii) $\alpha_1 + \cdots + \alpha_j = 1$.
Besides, for $n \geqslant m$ *we have*
(iii)

$$\alpha_i \leqslant \frac{C_{i,m}}{[n]_q^{m-i}}, \quad i = 1, \ldots, m$$

(iv)

$$\alpha_m = \lambda_{m,q}^{(n)}, \quad \alpha_{m-1} = \lambda_{m-1,q}^{(n)} \frac{1 + [2]_q + \cdots + [m-1]_q}{[n]_q}.$$

Proof. It was already noticed in the Preliminaries that $B_n(t^m, q; z)$ is a polynomial of degree $\min(m, n)$. The end-point interpolation property (3) implies that for $m \geqslant 1$, the free term of $B_n(t^m, q; z)$ equals 0. Therefore, (20) is justified.

(i) Representation (6) of q-Bernstein polynomials gives the following values of the coefficients in (20):

$$\alpha_i = \lambda_{i,q}^{(n)} f\left[0, \frac{1}{[n]_q}, \ldots, \frac{[i]_q}{[n]_q}\right], \quad i = 1, \ldots, m, \tag{21}$$

where $0 \leqslant \lambda_{i,q}^{(n)} \leqslant 1$ are given by (7).

Since for $f = t^m$:

$$f\left[0, \frac{1}{[n]_q}, \ldots, \frac{[i]_q}{[n]_q}\right] \geqslant 0,$$

the statement is proved.

(ii) This follows readily from (3), if we put $x = 1$ in (20).

(iii) Using (21) and (9), we get

$$\alpha_i \leqslant f\left[0, \frac{1}{[n]_q}, \ldots, \frac{[i]_q}{[n]_q}\right] = \frac{f^{(i)}(\xi_i)}{i!}, \quad \text{where } \xi_i \in \left(0, \frac{[i]_q}{[n]_q}\right).$$

Hence

$$\alpha_i \leqslant \binom{m}{i} \xi_i^{m-i} \leqslant \binom{m}{i} \left(\frac{[i]_q}{[n]_q}\right)^{m-i} =: \frac{C_{m,i}}{[n]_q^{m-i}},$$

as required.

(iv) Obviously,

$$f\left[0, \frac{1}{[n]_q}, \dots, \frac{[m]_q}{[n]_q}\right] = 1,$$

and, therefore $\alpha_m = \lambda_{m,q}^{(n)}$.

To calculate

$$f\left[0, \frac{1}{[n]_q}, \dots, \frac{[m-1]_q}{[n]_q}\right],$$

we use the representation (cf. [8, Chapter II, Section 2.7]):

$$f\left[0, \frac{1}{[n]_q}, \dots, \frac{[k]_q}{[n]_q}\right] = \frac{1}{2\pi i} \int_L \frac{f(\zeta)\, d\zeta}{\zeta\left(\zeta - \frac{1}{[n]_q}\right) \cdots \left(\zeta - \frac{[k]_q}{[n]_q}\right)},$$

where L is a contour around [0,1]. Hence for $f(\zeta) = \zeta^m$ we get

$$f\left[0, \frac{1}{[n]_q}, \dots, \frac{[m-1]_q}{[n]_q}\right] = \frac{1}{2\pi i} \int_L \frac{\zeta^{m-1}\, d\zeta}{\left(\zeta - \frac{1}{[n]_q}\right) \cdots \left(\zeta - \frac{[m-1]_q}{[n]_q}\right)}.$$

Direct calculation of the integral implies

$$f\left[0, \frac{1}{[n]_q}, \dots, \frac{[m-1]_q}{[n]_q}\right] = \frac{1 + [2]_q + \cdots + [m-1]_q}{[n]_q}$$

and (iv) is proved. $\square$

Proof of Theorem 4. For $m = 0, 1$ there is nothing to prove, because by (11) q-Bernstein polynomials leave invariant linear functions.

For $m \geqslant 2$ using (iii) and (iv) of Lemma 3, we get

$$\begin{aligned}
&\lim_{n\to\infty} [n]_q\{B_n(t^m, q; z) - z^m\} \\
&\quad = \lim_{n\to\infty} [n]_q\{\alpha_m z^m + \alpha_{m-1} z^{m-1} - z^m\} \\
&\quad = \lim_{n\to\infty} [n]_q\left\{(\lambda_{m,q}^{(n)} - 1)z^m + \frac{1 + [2]_q + \cdots + [m-1]_q}{[n]_q} z^{m-1}\right\} \\
&\quad = (1 + [2]_q + \cdots + [m-1]_q)z^{m-1} \\
&\qquad + z^m \lim_{n\to\infty} [n]_q\left\{\left(1 - \frac{1}{[n]_q}\right)\ldots\left(1 - \frac{[m-1]_q}{[n]_q}\right) - 1\right\} \\
&\quad = (1 + [2]_q + \cdots + [m-1]_q)(z^{m-1} - z^m). \qquad \square
\end{aligned}$$

Proof of Theorem 5. For $m = 0, 1$ the statement is obvious.

First we consider the case $n \geqslant m \geqslant 2$. Applying Lemma 3, we get for $|z| \leqslant R$, $R > 1$:

$$\begin{aligned}
|B_n(t^m, q; z) - z^m| &= \left|\sum_{k=1}^{m-1} \alpha_k z^k + (1 - \lambda_{m,q}^{(n)})z^m\right| \\
&\leqslant \left(\sum_{k=1}^{m-1} \alpha_k + (1 - \lambda_{m,q}^{(n)})\right)R^m = 2(1 - \lambda_{m,q}^{(n)})R^m. \qquad (22)
\end{aligned}$$

Now, by (7)

$$\begin{aligned}
1 - \lambda_{m,q}^{(n)} &= 1 - \left(1 - \frac{1}{[n]_q}\right)\cdots\left(1 - \frac{[m-1]_q}{[n]_q}\right) \\
&\leqslant 1 - \left(1 - \frac{[m-1]_q}{[n]_q}\right)^{m-1} \leqslant (m-1)\frac{[m-1]_q}{[n]_q}.
\end{aligned}$$

Using (22), we get that for $n \geqslant m$,

$$|B_n(t^m, q; z) - z^m| \leqslant 2(m-1)\frac{[m-1]_q}{[n]_q} R^m.$$

To complete the proof, we note that statements (i) and (ii) of Lemma 3 yield that $|B_n(t^m, q; z) - z^m| \leqslant 2R^m$. Therefore, the estimate is also true for $n < m$. $\square$

Proof of Theorem 6. Let $f(z) = \sum_{m=0}^{\infty} a_m z^m$ be a function analytic in a disk D_R, $R > q$. Evidently,

$$B_n(f, q; z) = \sum_{m=0}^{\infty} a_m B_n(t^m, q; z) \quad \text{for } |z| \leqslant R.$$

Hence applying Corollary 6 of Theorem 5, we have for $|z| \leqslant 1$:

$$|B_n(f,q;z) - f(z)| \leqslant \sum_{m=0}^{\infty} |a_m||B_n(t^m,q;z) - z^m| \leqslant \sum_{m=0}^{\infty} \frac{2|a_m|mq^m}{[n]_q(q-1)} =: \frac{C_{f,q}}{[n]_q},$$

because $\sum_{m=0}^{\infty} |a_m|mq^m < \infty$. $\square$

Proof of Theorem 7. First, we prove that for all $q \geqslant 1$ and all $m = 0, 1, \ldots$ $n = 1, 2, \ldots$ the following estimate holds uniformly with respect to q:

$$|B_n(t^m,q;z) - z^m| \leqslant \frac{2m^2}{n} \quad \text{for } |z| \leqslant 1.$$

If $n < m$, the inequality is true, because $|B_n(t^m,q;z) - z^m| \leqslant 2$ for $|z| \leqslant 1$. For $n \geqslant m$, we have by (22)

$$|B_n(t^m,q;z) - z^m| \leqslant 2(1 - \lambda_{m,q}^{(n)}) \quad \text{for } |z| \leqslant 1.$$

If $q \geqslant 1$, then

$$\frac{[j]_q}{[n]_q} \leqslant \frac{j}{n} \quad \text{for } j = 0, 1, \ldots, n,$$

and hence

$$\lambda_{m,q}^{(n)} = \left(1 - \frac{1}{[n]_q}\right) \cdots \left(1 - \frac{[m-1]_q}{[n]_q}\right) \geqslant \left(1 - \frac{1}{n}\right) \cdots \left(1 - \frac{m-1}{n}\right) = \lambda_{m,1}^{(n)}.$$

Therefore, for all $q \geqslant 1$ we get

$$\begin{aligned} |B_n(t^m,q;z) - z^m| &\leqslant 2(1 - \lambda_{m,1}^{(n)}) \\ &\leqslant 2\left[1 - \left(1 - \frac{m-1}{n}\right)^{m-1}\right] \leqslant 2\frac{(m-1)^2}{n} \leqslant 2\frac{m^2}{n} \quad \text{for } |z| \leqslant 1. \end{aligned}$$

Now, let $f(z) = \sum_{m=0}^{\infty} a_m z^m$ be a function analytic in a disk D_R, $R > 1$. Then for any $q \geqslant 1$,

$$|B_n(f,q;z) - f(z)| \leqslant \sum_{m=0}^{\infty} |a_m||B_n(t^m,q;z) - z^m| \leqslant \sum_{m=0}^{\infty} 2\frac{|a_m|m^2}{n} =: \frac{C_f}{n},$$

since $\sum_{m=0}^{\infty} |a_m|m^2 < \infty$. Clearly, C_f does not depend on q. $\square$

Remark. The statement remains true if $f(z) = \sum_{m=0}^{\infty} a_m z^m$ is a function analytic in the open unit disk and $\sum_{m=0}^{\infty} |a_m|m^2 < \infty$.

7. Iterates of q-Bernstein polynomials

We recall that the *q-Bernstein operator* $B_{n,q} : C[0,1] \to \mathscr{P}_n$ is defined by

$$B_{n,q} : f \mapsto B_n(f, q; x) = \sum_{k=0}^{n} f\left(\frac{[k]_q}{[n]_q}\right) p_{nk}(q; x),$$

where $p_{nk}(q; x)$ are given by (12).

For $q \in (0,1)$ equality (16) defines the limit operator $B_{\infty,q}$ on $C[0,1]$ as

$$B_{\infty,q} : f \mapsto B_\infty(f, q; x) = \begin{cases} \sum_{k=0}^{\infty} f(1-q^k) p_{\infty k}(q; x) & \text{if } x \in [0,1), \\ f(1) & \text{if } x = 1, \end{cases} \tag{23}$$

where entire functions $p_{\infty,k}$ are given by (14).

It can be readily seen that for $q \in (0,1)$, both polynomials $p_{nk}(q; x)$ and entire functions $p_{\infty k}(q; x)$ are non-negative on the interval $[0,1]$. Therefore, we get from (13) and (15) that

$$||B_{n,q}|| = ||B_{\infty,q}|| = 1 \quad \text{for } q \in (0,1). \tag{24}$$

By L we denote the operator of linear interpolation at 0 and 1, i.e.,

$$L(f; x) := (1-x)f(0) + xf(1).$$

Theorem 8. *Let $q \in (0,1)$ and $\{j_n\}$ be a sequence of positive integers such that $j_n \to \infty$. Then for any $f \in C[0,1]$,*

$$B_n^{j_n}(f, q; x) \rightrightarrows L(f; x) \quad \textit{for } x \in [0,1] \textit{ as } n \to \infty.$$

The following theorem describes the behavior of iterates of the limit operator $B_{\infty,q}$.

Theorem 9. *Let $q \in (0,1)$, and the operator $B_{\infty,q}$ be defined by (23). If $\{j_n\}$ is a sequence of positive integers such that $j_n \to \infty$, then for any $f \in C[0,1]$,*

$$B_\infty^{j_n}(f, q; x) \rightrightarrows L(f; x) \quad \textit{for } x \in [0,1] \textit{ as } n \to \infty.$$

The statement below (proved in [5]) follows from Theorem 9 immediately.

Corollary 8. *Let $q \in (0,1)$. Then $B_{\infty,q}(f) = f$ if and only if $f = L(f)$, i.e. f is a linear function.*

For $q \in (1, \infty)$ we restrict ourselves to the case when f is a polynomial. This is because in contrast to the case $q \in (0,1]$, the sequence $\{B_n(f, q; x)\}$ may be divergent even for an infinitely differentiable function f (cf. Theorem 2.) However, behavior of the operators $B_{n,q}$ for $q \in (1, \infty)$ on the space of polynomials $\mathscr{P} = \bigcup_{m=0}^{\infty} \mathscr{P}_m$ is rather similar to the classical case. In particular, for any $p \in \mathcal{P}$ the sequence $\{B_n(p, q; x)\}$ converges to p uniformly on [0,1]. Behavior of iterates $B_{n,q}^{j_n}$ on $\mathcal{P}$ resembles the situation with $q = 1$. More precisely, the following statement holds.

Theorem 10. *Let* $q \in (1, \infty)$ *and* $\{j_n\}$ *be a sequence of positive integers such that* $j_n/[n]_q \to t$ *as* $n \to \infty$. *Then for any polynomial* p *and any* $0 \leqslant t \leqslant \infty$ *the sequence* $\{B_n^{j_n}(p, q; x)\}$ *converges uniformly on* $[0, 1]$. *In particular, for* $t = 0$,

$$B_n^{j_n}(p, q; x) \rightrightarrows p(x) \quad \text{for } x \in [0, 1],$$

and for $t = \infty$,

$$B_n^{j_n}(p, q; x) \rightrightarrows L(p; x) \quad \text{for } x \in [0, 1].$$

We omit the proof of Theorem 10 since it repeats verbatim the reasoning of [3], where the classical case $q = 1$ was considered.

8. Some auxiliary results

Lemma 4. *For all* $q > 0$ *the following identity holds*:

$$B_n(t^m, q; x) = \frac{x}{[n]_q^{m-1}} \sum_{j=0}^{m-1} \binom{m-1}{j} ([n]_q - 1)^j B_{n-1}(t^j, q; x),$$
$$n = 2, 3, \ldots; \; m = 1, 2, \ldots \; . \tag{25}$$

Proof. Let $p_{n,k}(q; x)$ be defined by (12). Then

$$\begin{aligned}
B_n(t^m, q; x) &= \sum_{k=0}^{n} \left(\frac{[k]_q}{[n]_q} \right)^m p_{nk}(q; x) \\
&= \sum_{k=1}^{n} \left(\frac{[k]_q}{[n]_q} \right)^{m-1} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q x^k \prod_{s=0}^{n-k-1} (1 - q^s x) \\
&= \frac{x}{[n]_q^{m-1}} \sum_{k=0}^{n-1} [k+1]_q^{m-1} p_{n-1,k}(q; x) \\
&= \frac{x}{[n]_q^{m-1}} \sum_{k=0}^{n-1} (1 + q[k]_q)^{m-1} p_{n-1,k}(q; x) \\
&= \frac{x}{[n]_q^{m-1}} \sum_{j=0}^{m-1} \binom{m-1}{j} (q[n-1]_q)^j \left(\sum_{k=0}^{n-1} \left(\frac{[k]_q}{[n-1]_q} \right)^j p_{n-1,k}(q; x) \right) \\
&= \frac{x}{[n]_q^{m-1}} \sum_{j=0}^{m-1} \binom{m-1}{j} ([n]_q - 1)^j B_{n-1}(t^j, q; x). \qquad \square
\end{aligned}$$

Lemma 5. *For all $q>0$ the operator $B_{n,q}$ has $(n+1)$ linearly independent monic eigenvectors $p_m^{(n)}(x)$, $\deg p_m^{(n)}(x)=m$, $(m=0,1,\ldots,n)$, corresponding to the eigenvalues*

$$\lambda_{0,q}^{(n)}=\lambda_{1,q}^{(n)}=1,$$

$$\lambda_{m,q}^{(n)}=\left(1-\frac{[1]_q}{[n]_q}\right)\left(1-\frac{[2]_q}{[n]_q}\right)\cdots\left(1-\frac{[m-1]_q}{[n]_q}\right),\quad \text{for } m=2,\ldots,n. \tag{26}$$

Remark. For $q=1$, (26) coincides with formula (2.5) in [3].

Proof. For $m=0,1$ the statement is obvious due to (11). For $n\geqslant m\geqslant 2$, using Lemma 3 we write

$$B_n(t^m,q;x)=\lambda_{m,q}^{(n)}x^m+P_{m-1}^{(n)}(x), \tag{27}$$

where $P_{m-1}^{(n)}(x)\in\mathscr{P}_{m-1}$ and $\lambda_{m,q}^{(n)}$ are given by (26).

To find an eigenvector $p_m^{(n)}\in\mathscr{P}_m$ of the operator $B_{n,q}$, we write $p_m^{(n)}=x^m+a_{m-1}x^{m-1}+\cdots+a_1x$ and solve a linear system in unknowns $a_1,\ldots,a_{m-1}$:

$$B_{n,q}(x^m+a_{m-1}x^{m-1}+\cdots+a_1x)=\lambda_{m,q}^{(n)}(x^m+a_{m-1}x^{m-1}+\cdots+a_1x).$$

After we apply $B_{n,q}$ in the left-hand side and equate the coefficients of x^s $(s=1,\ldots m-1)$, we get a triangular system whose determinant equals

$$(\lambda_{m-1,q}^{(n)}-\lambda_{m,q}^{(n)})(\lambda_{m-2,q}^{(n)}-\lambda_{m,q}^{(n)})\ldots(\lambda_{1,q}^{(n)}-\lambda_{m,q}^{(n)})\neq 0.$$

Hence there exists a unique monic polynomial of degree $2\leqslant m\leqslant n$ which is an eigenvector of $B_{n,q}$ with the eigenvalue $\lambda_{m,q}^{(n)}$. $\square$

Corollary 9. *For $2\leqslant m\leqslant n$, the operator $\lambda_{m,q}^{(n)}I-B_{n,q}$, where I is the identity operator, is invertible on $\mathscr{P}_{m-1}$.*

Lemma 6. *The following equality holds*:

$$\lim_{n\to\infty}\lambda_{m,q}^{(n)}=\begin{cases} q^{\frac{m(m-1)}{2}}\ (m=0,1,2,\ldots) & \text{if } q\in(0,1),\\ 1 & \text{if } q\in[1,\infty).\end{cases}$$

Proof. The statement follows from formula (26) after we notice that

$$\lim_{n\to\infty}\left(1-\frac{[j]_q}{[n]_q}\right)=\begin{cases} q^j\ (j=0,1,2,\ldots) & \text{if } q\in(0,1),\\ 1 & \text{if } q\in[1,\infty).\end{cases}\qquad\square$$

Lemma 7. *Let $q \in (0,1)$. Then for every $m = 0,1,2\ldots$ the operator $B_{\infty,q}$ has an eigenvector $p_m(x)$ which is a monic polynomial of degree m, corresponding to the eigenvalue $\lambda_{m,q} = q^{m(m-1)/2}$.*

Proof. For $m = 0,1$ the statement follows immediately from (17).

Taking the limit as $n \to \infty$ in (25) and noting that for $q \in (0,1)$ one has

$$\frac{([n]_q - 1)^j}{[n]_q^{m-1}} \to q^j (1-q)^{m-j-1},$$

we get

$$B_\infty(t^m, q; x) = x \sum_{j=0}^{m-1} \binom{m-1}{j} q^j (1-q)^{m-j-1} B_\infty(t^j, q, x).$$

Hence the coefficient $\lambda_{m,q}$ of x^m in $B_\infty(t^m, q; x)$ equals $q^{m-1}\lambda_{m-1,q}$, and recursively,

$$\lambda_{m,q} = q^{m-1} q^{m-2} \ldots q \lambda_{1q} = q^{m(m-1)/2}.$$

We have shown that

$$B_\infty(t^m, q; x) = \lambda_{m,q} x^m + Q_{m-1}, \quad Q_{m-1} \in \mathscr{P}_{m-1}.$$

The statement now follows from considering the equations

$$B_{\infty,q}(p_m(x)) = \lambda_{m,q} p_m(x), \quad m = 2, 3, \ldots \quad . \qquad \square$$

Corollary 10. *For $m \geqslant 2$, the operator $\lambda_{m,q} I - B_{\infty,q}$ is invertible on $\mathscr{P}_{m-1}$.*

9. Proofs of Theorems 8 and 9

In this section $\rightrightarrows$ means uniform convergence on $[0,1]$.

Proof of Theorem 8. Because of (3) it suffices to prove that $B_{n,q}^{j_n}(f) \rightrightarrows ax + b$ for some a and b as $n \to \infty$.

(1) First we consider the case $f = x^m$.

We will use induction on m. For $m = 0,1$ the statement is obvious due to (11).

Assume that $B_{n,q}^{j_n}(x^t) \rightrightarrows \varphi_t \in \mathscr{P}_1$ for $t = 0, 1, \ldots, m-1$. Consider

$$B_{n,q}(x^m) = \lambda_{m,q}^{(n)} x^m + P_{m-1}^{(n)}, \tag{28}$$

where $\lambda_{m,q}^{(n)}$ is given by (26), and $P_{m-1}^{(n)} \in \mathscr{P}_{m-1}$. Then

$$B_{n,q}^{j_n}(x^m) = (\lambda_{m,q}^{(n)})^{j_n} x^m + [(\lambda_{m,q}^{(n)})^{j_n-1} I + (\lambda_{m,q}^{(n)})^{j_n-2} B_{n,q} + \cdots + B_{n,q}^{j_n-1}](P_{m-1}^{(n)}),$$

where I denotes the identity operator. It follows from Lemma 6 that

$$(\lambda_{m,q}^{(n)})^{j_n} \to 0 \quad \text{as } n \to \infty.$$

The expression in the brackets is a linear operator on the space $\mathscr{P}_{m-1}$.

Consider the sequence of polynomials in $\mathscr{P}_{m-1}$,

$$y_{m-1}^{(n)} := [(\lambda_{m,q}^{(n)})^{j_n-1} I + (\lambda_{m,q}^{(n)})^{j_n-2} B_{n,q} + \cdots + B_{n,q}^{j_n-1}](P_{m-1}^{(n)}). \tag{29}$$

Then

$$(\lambda_{m,q}^{(n)} I - B_{n,q}) y_{m-1}^{(n)} = (\lambda_{m,q}^{(n)})^{j_n} P_{m-1}^{(n)} - B_{n,q}^{j_n} P_{m-1}^{(n)}.$$

It follows from (24) and (28) that $||P_{m-1}^{(n)}|| \leqslant 2$. Since $(\lambda_{m,q}^{(n)})^{j_n} \to 0$ as $n \to \infty$, we have

$$(\lambda_{m,q}^{(n)})^{j_n} P_{m-1}^{(n)} \rightrightarrows 0 \quad \text{as } n \to \infty.$$

It can be readily seen from (28) and Lemma 6 that

$$P_{m-1}^{(n)}(x) \rightrightarrows B_{\infty,q}(x^m) - q^{m(m-1)/2} x^m =: Q_{m-1}(x) \in \mathscr{P}_{m-1} \quad \text{as } n \to \infty,$$

i.e.

$$P_{m-1}^{(n)}(x) = Q_{m-1}(x) + \delta_n(x),$$

where $Q_{m-1} \in \mathscr{P}_{m-1}$, and $\delta_n(x) \rightrightarrows 0$ as $n \to \infty$.

Thus,

$$B_{n,q}^{j_n}(P_{m-1}^{(n)}) = B_{n,q}^{j_n}(Q_{m-1}) + B_{n,q}^{j_n}(\delta_n),$$

where $||B_{n,q}^{j_n}(\delta_n)|| \leqslant ||\delta_n||$, because of (24). This means that $B_{n,q}^{j_n}(\delta_n) \rightrightarrows 0$ as $n \to \infty$.

By the induction assumption

$$B_{n,q}^{j_n}(Q_{m-1}) \rightrightarrows cx + d \in \mathscr{P}_1 \quad \text{as } n \to \infty.$$

Therefore,

$$(\lambda_{m,q}^{(n)} I - B_{n,q}) y_{m-1}^{(n)} \rightrightarrows cx + d \quad \text{as } n \to \infty$$

or

$$(\lambda_{m,q}^{(n)} I - B_{n,q}) y_{m-1}^{(n)} = cx + d + \omega_n(x),$$

where $\omega_n(x) \rightrightarrows 0$ as $n \to \infty$.

By Corollary 8 the operators $\lambda_{m,q}^{(n)} I - B_{n,q}$ are invertible on $\mathscr{P}_{m-1}$ for $n \geqslant m$ and

$$\lim_{n \to \infty} (\lambda_{m,q}^{(n)} I - B_{n,q}) = q^{\frac{m(m-1)}{2}} I - B_{\infty,q} =: A_{\infty,q},$$

where by Corollary 10 $A_{\infty,q}$ is also invertible on $\mathscr{P}_{m-1}$. Hence

$$(\lambda_{m,q}^{(n)} I - B_{n,q})^{-1} \to A_{\infty,q}^{-1} \quad \text{as } n \to \infty$$

and it follows that

$$||(\lambda_{m,q}^{(n)} I - B_{n,q})^{-1}|| \leqslant M \quad \text{for some } M > 0.$$

Therefore,

$$y_{m-1}^{(n)} = (\lambda_{m,q}^{(n)} I - B_{n,q})^{-1}(cx+d) + (\lambda_{m,q}^{(n)} I - B_{n,q})^{-1}(\omega_n).$$

Since $||(\lambda_{m,q}^{(n)} I - B_{n,q})^{-1}(\omega_n)|| \leqslant M||\omega_n|| \to 0$ as $n \to \infty$, and $(\lambda_{m,q}^{(n)} I - B_{n,q})^{-1} \to A_{\infty,q}$ as $n \to \infty$, we conclude that

$$y_{m-1}^{(n)} \rightrightarrows A_{\infty,q}^{-1}(cx+d) := ax+b \in \mathscr{P}_1.$$

Thus, $B_{n,q}^{j_n}(x^m) \rightrightarrows ax+b$.

The induction is completed and it follows that for any polynomial p,

$$B_n^{j_n}(p,q;x) \rightrightarrows L(p;x) \quad \text{for } x \in [0,1] \text{ as } n \to \infty.$$

(2) Let $f \in C[0,1]$, and let $\varepsilon > 0$ be given. Then $f(x) = p(x) + \delta(x)$, where $p \in \mathscr{P}$, and $||\delta(x)|| < \varepsilon$. We have

$$B_{n,q}^{j_n}(f) = B_{n,q}^{j_n}(p) + B_{n,q}^{j_n}(\delta).$$

Since $B_{n,q}^{j_n}(p) \rightrightarrows L(p)$, there exists $n_0 \in \mathbf{N}$ such that $||B_n^{j_n}(p) - L(p)|| < \varepsilon$ for all $n > n_0$. Obviously, $||L(\delta)|| \leqslant ||\delta|| < \varepsilon$, and finally we obtain

$$||B_{n,q}^{j_n}(f) - L(f)|| \leqslant ||B_{n,q}^{j_n}(p) - L(p)|| + ||B_n^{j_n}(\delta)|| + ||\delta|| < 3\varepsilon \quad \text{for all } n > n_0.$$

Thus, $B_n^{j_n}(f,q;x) \rightrightarrows L(f;x)$ for $x \in [0,1]$ as $n \to \infty$. □

Proof of Theorem 9. (1) First we prove the statement in the case $f \in \mathscr{P}_m$. For $f \in \mathscr{P}_m$ by Lemma 7 we have

$$f = \alpha_0 p_0 + \alpha_1 p_1 + \cdots + \alpha_m p_m,$$

where $p_0, p_1, \ldots, p_m$ are eigenvectors of $B_{\infty,q}$ corresponding to the eigenvalues $\lambda_{0,q}$, $\lambda_{1,q}, \ldots, \lambda_{m,q}$. Obviously,

$$B_{\infty,q}^{j_n}(f) = \alpha_0 \lambda_{0,q}^{j_n} p_0 + \alpha_1 \lambda_{1,q}^{j_n} p_1 + \cdots + \alpha_m \lambda_{m,q}^{j_n} p_m.$$

Since $\lambda_{0,q} = \lambda_{1,q} = 1$, $\lambda_{i,q} \in (0,1)$ for $i \geqslant 2$, we obtain

$$B_{\infty,q}^{j_n}(f) \rightrightarrows \alpha_0 p_0 + \alpha_1 p_1 \in \mathscr{P}_1.$$

Taking into account (3), we derive the statement.

(2) For $f \in C[0,1]$, the statement follows from the density of the set of polynomials in $C[0,1]$ and the fact that $||B_{\infty,q}|| = 1$ (cf. (24)). □

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References

[1] G.E. Andrews, The Theory of Partitions, Addison-Wesley, Reading, MA, 1976.

[2] S.N. Bernstein, Démonstration du théorème de Weierstrass fondée sur la calcul des probabilités, Comm. Soc. Math. Charkow Sér. 2 t. 13 (1912) 1–2.

[3] S. Cooper, S. Waldron, The eigenstructure of the Bernstein operator, J. Approx. Theory 105 (2000) 133–165.

[4] T.N.T. Goodman, H. Oruç, G.M. Phillips, Convexity and generalized Bernstein polynomials, Proc. Edinburgh Math. Soc. 42 (1) (1999) 179–190.

[5] A. Il'inskii, S. Ostrovska, Convergence of generalized Bernstein polynomials, J. Approx. Theory 116 (2002) 100–112.

[6] R.P. Kelisky, T.J. Rivlin, Iterates of Bernstein polynomials, Pacific J. Math. 21 (1967) 511–520.

[7] X. Li, P. Mikusiński, H. Sherwood, M.D. Taylor, On approximation of copulas, in: V. Benes, J. Stepan (Eds.), Distributions with Given Marginals and Moment Problem, Kluwer Academic Publishers, Dordrecht, 1997.

[8] G.G. Lorentz, Bernstein Polynomials, Chelsea, New York, 1986.

[9] A. Lupaş, A q-analogue of the Bernstein operator, University of Cluj-Napoca, Seminar on Numerical and Statistical Calculus, preprint No. 9, 1987.

[10] C. Micchelli, The saturation class and iterates of the Bernstein polynomials, J. Approx. Theory 8 (1973) 1–18.

[11] H. Oruç, G.M. Phillips, A generalization of Bernstein polynomials, Proc. Edinburgh Math. Soc. 42 (2) (1999) 403–413.

[12] H. Oruç, N. Tuncer, On the convergence and iterates of q-Bernstein polynomials, J. Approx. Theory 117 (2002) 301–313.

[13] S. Petrone, Random Bernstein polynomials, Scand. J. Statist. 26 (3) (1999) 373–393.

[14] G.M. Phillips, On generalized Bernstein polynomials, in: D.F. Griffits, G.A. Watson (Eds.), Numerical Analysis: A.R. Mitchell 75th Birthday Volume, World Science, Singapore, 1996, pp. 263–269.

[15] G.M. Phillips, Bernstein polynomials based on the q-integers, Ann. Numer. Math. 4 (1997) 511–518.

[16] G.M. Phillips, A de Casteljau algorithm for generalized Bernstein polynomials, BIT 37 (1) (1997) 232–236.

[17] G.M. Phillips, A generalization of the Bernstein polynomials based on the q-integers, ANZIAM J. 42 (2000) 79–86.

[18] E.C. Titchmarsh, Theory of Functions, Oxford University Press, Oxford, 1986.

[19] V.S. Videnskii, Bernstein Polynomials, Leningrad State Pedagocical University, Leningrad, 1990 (Russian).